Exercise 3

Evaluate the line integral, where C is the given curve.

$$\int_C xy^4\,ds, \quad C \text{ is the right half of the circle } x^2+y^2=16$$

Solution

Parameterize the right half of the circle by $x = 4\cos t$ and $y = 4\sin t$ with $-\pi/2 \le t \le \pi/2$. With this parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C xy^4\,ds &= \int_{-\pi/2}^{\pi/2} x(t)[y(t)]^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4 \sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt \\
&= \int_{-\pi/2}^{\pi/2} (4\cos t)(256\sin^4 t)\sqrt{16\sin^2 t + 16\cos^2 t}\,dt \\
&= \int_{-\pi/2}^{\pi/2} (4\cos t)(256\sin^4 t)4\sqrt{\sin^2 t + \cos^2 t}\,dt \\
&= 4096\int_{-\pi/2}^{\pi/2} \cos t\sin^4 t\,dt.
\end{aligned}$$

Make the following substitution.

$$u = \sin t$$

$$du = \cos t\,dt$$

Therefore,

$$\begin{aligned}
\int_C xy^4\,ds &= 4096\int_{\sin(-\pi/2)}^{\sin(\pi/2)} u^4\,du \\
&= 4096\int_{-1}^{1} u^4\,du \\
&= 4096\left(\frac{1}{5}u^5\right)\bigg|_{-1}^{1} \\
&= \frac{4096}{5}[1^5 - (-1)^5] \\
&= \frac{8192}{5} \\
&\approx 1638.4.
\end{aligned}$$